ELSEVIER

Discrete Mathematics 146 (1995) 59–67

DISCRETE MATHEMATICS

A characterization of some $\{3v_{\mu+1}, 3v_\mu; k-1, q\}$-minihypers and some $[n, k, q^{k-1} - 3q^\mu; q]$-codes ($k \geqslant 3$, $q \geqslant 5$, $1 \leqslant \mu < k-1$) meeting the Griesmer bound

Noboru Hamada[a], Tor Helleseth[b,*]

[a] *Department of Applied Mathematics, Osaka Women's University, Daisen-cho, Sakai, Osaka 590, Japan*
[b] *Department of Informatics, University of Bergen, Høyteknologisenteret, N-5020 Bergen, Norway*

Received 18 May 1993; revised 1 March 1994

Abstract

For any $[n, k, d; q]$-code the Griesmer bound says that $n \geqslant \sum_{i=0}^{k-1} \lceil d/q^i \rceil$. The purpose of this paper is to characterize all $[n, k, q^{k-1} - 3q^\mu; q]$-codes meeting the Griesmer bound in the case where $k \geqslant 3$, $q \geqslant 5$ and $1 \leqslant \mu < k-1$. It is shown that all such codes have a generator matrix whose columns correspond to all points in $\mathrm{PG}(k-1, q)$ except for the points in a disjoint union of three μ-flats in $\mathrm{PG}(k-1, q)$.

1. Introduction

Let $V(n, q)$ be an n-dimensional vector space consisting of row vectors over the Galois field $\mathrm{GF}(q)$ where $n \geqslant 3$ and q is a prime power. If C is a k-dimensional subspace in $V(n, q)$ such that every nonzero vector in C has a Hamming weight of at least d, then C is called an $[n, k, d; q]$-code. Let $n_q(k, d)$ denote the smallest value of n for which there exists an $[n, k, d; q]$-code. An $[n_q(k, d), k, d; q]$-code is therefore optimal in the sense that no shorter code exists with the same k, d and q. It is well known (cf. [1, 12]) that if there exists an $[n, k, d; q]$-code, then

$$n \geqslant \sum_{i=0}^{k-1} \left\lceil \frac{d}{q^i} \right\rceil, \tag{1.1}$$

where $\lceil x \rceil$ denotes the smallest integer $\geqslant x$. This bound (called the Griesmer bound) shows that if an $[n, k, d; q]$-code meeting the Griesmer bound exists, then this code is optimal. Hence, we shall consider the following problem.

* Corresponding author.

0012-365X/95/$09.50
SSDI 0012-365X(94)00157-X

Problem 1.1. (1) Find a necessary and sufficient condition for integers k, d and q such that there exists an $[n, k, d; q]$-code meeting the Griesmer bound.

(2) Characterize up to equivalence (cf. Definition A.1) all $[n, k, d; q]$-codes meeting the Griesmer bound for given values of k, d and q when such $[n, k, d; q]$-codes exist.

In the case $q = 2$, and $1 \leqslant d \leqslant 2^{k-1}$, Problem 1.1 was solved completely by Helleseth [11]. Hence, we restrict ourselves to the case $q \geqslant 3$, $k \geqslant 3$, and $1 \leqslant d < q^{k-1}$. In this case d can be expressed uniquely as follows:

$$d = q^{k-1} - \sum_{i=1}^{h} q^{\lambda_i} \tag{1.2}$$

using some integers k, q, h and λ_i $(i = 1, 2, \ldots, h)$ such that (a) $1 \leqslant h \leqslant (k-1)(q-1)$, $0 \leqslant \lambda_1 \leqslant \lambda_2 \leqslant \cdots \leqslant \lambda_h < k-1$ and (b) at most $q-1$ of the λ_i's take the same value and the Griesmer bound (1.1) can be expressed as follows:

$$n \geqslant v_k - \sum_{i=1}^{h} v_{\lambda_i+1}, \tag{1.3}$$

where $v_l = (q^l - 1)/(q-1)$ for any integer $l \geqslant 0$.

Let $\mathrm{PG}(t, q)$ be a finite projective geometry of dimension t over $\mathrm{GF}(q)$. An r-flat is a subspace of projective dimension r of $\mathrm{PG}(t, q)$.

Definition 1.1. Let F be a set of f points in $\mathrm{PG}(t, q)$ where $t \geqslant 2$ and $f \geqslant 1$. If $|F \cap H| \geqslant m$ for every hyperplane H in $\mathrm{PG}(t, q)$ and $|F \cap H| = m$ for some hyperplane H in $\mathrm{PG}(t, q)$, then F is called an $\{f, m; t, q\}$-minihyper where $m \geqslant 0$ and $|A|$ denotes the number of elements in the set A. In the special case $t = 2$ and $m \geqslant 2$, an $\{f, m; 2, q\}$-minihyper F is also called an m-blocking set if F contains no 1-flat in $\mathrm{PG}(2, q)$.

Hamada [3] showed that in the case $d = q^{k-1} - \sum_{i=1}^{h} q^{\lambda_i}$, there is a one-to-one correspondence between the set of all nonequivalent $[n, k, d; q]$-codes meeting the Griesmer bound and the set of all $\{\sum_{i=1}^{h} v_{\lambda_i+1}, \sum_{i=1}^{h} v_{\lambda_i};\ k-1, q\}$-minihypers (cf. Corollary A.1). Hence, in order to solve Problem 1.1, it is sufficient to solve the following problem.

Problem 1.2. (1) Find a necessary and sufficient condition for integers t, q, h and λ_i $(i = 1, 2, \ldots, h)$ such that there exists a $\{\sum_{i=1}^{h} v_{\lambda_i+1}, \sum_{i=1}^{h} v_{\lambda_i};\ t, q\}$-minihyper where $t \geqslant 2$, $q \geqslant 3$, $1 \leqslant h \leqslant t(q-1)$, $0 \leqslant \lambda_1 \leqslant \lambda_2 \leqslant \cdots \leqslant \lambda_h < t$ and at most $q-1$ of the λ_i's take the same value.

(2) Characterize all $\{\sum_{i=1}^{h} v_{\lambda_i+1}, \sum_{i=1}^{h} v_{\lambda_i};\ t, q\}$-minihypers when such minihypers exist.

Definition 1.2. Let $\mathscr{F}(\lambda_1, \lambda_2, \ldots, \lambda_h;\ t, q)$ denote the family of all unions $\bigcup_{i=1}^{h} V_i$ of a λ_1-flat V_1, a λ_2-flat $V_2, \ldots,$ a λ_h-flat V_h in $\mathrm{PG}(t, q)$ which are mutually disjoint, where $1 \leqslant h \leqslant t(q-1)$, $0 \leqslant \lambda_1 \leqslant \lambda_2 \leqslant \cdots \leqslant \lambda_h < t$ and at most $q-1$ of the λ_i's take the same value.

Remark 1.1. (1) It is known (cf. [10]) that in the case $h \geqslant 2$, $\mathscr{F}(\lambda_1, \lambda_2, \ldots, \lambda_h;\ t, q) \neq \emptyset$ if and only if $t \geqslant \lambda_{h-1} + \lambda_h + 1$.

(2) It is known (cf. [3]) that if $F \in \mathscr{F}(\lambda_1, \lambda_2, \ldots, \lambda_h;\ t, q)$ in the case $h \geqslant 2$ and $t \geqslant \lambda_{h-1} + \lambda_h + 1$, then F is a $\{\sum_{i=1}^{h} v_{\lambda_i+1}, \sum_{i=1}^{h} v_{\lambda_i};\ t, q\}$-minihyper.

For the case $h = 1$, Problem 1.2 was solved by Tamari [13, 14]. For the case $h = 2$, Problem 1.2 was solved by Hamada [2, 5]. For the case $h = 3$, Problem 1.2 was solved by Hamada et al. [2–9] except for the case $1 \leqslant \lambda_1 = \lambda_2 = \lambda_3 < t$ and $q \geqslant 5$. The purpose of this paper is to give a solution of Problem 1.2 for this remaining case, i.e., to prove the following theorem.

Theorem 1.1. *Let t, μ and q be any integers such that $1 < \mu < t$ and $q \geqslant 5$.*

(1) *There exists a $\{3v_{\mu+1}, 3v_\mu;\ t, q\}$-minihyper if and only if $t \geqslant 2\mu + 1$.*

(2) *In the case $t \geqslant 2\mu + 1$, F is a $\{3v_{\mu+1}, 3v_\mu;\ t, q\}$-minihyper if and only if $F \in \mathscr{F}(\mu, \mu, \mu;\ t, q)$, i.e., F is a union of three μ-flats in $\mathrm{PG}(t, q)$ which are mutually disjoint.*

From Theorem 1.1 and Corollary A.1, we have the following corollary.

Corollary 1.1. *Let k, μ and q be any integers such that $1 \leqslant \mu < k - 1$ and $q \geqslant 5$.*

(1) *There exists a $[v_k - 3v_{\mu+1}, k, q^{k-1} - 3q^\mu;\ q]$-code meeting the Griesmer bound (1.3) if and only if $k \geqslant 2\mu + 2$.*

(2) *In the case $k \geqslant 2\mu + 2$, C is a $[v_k - 3v_{\mu+1}, k, q^{k-1} - 3q^\mu;\ q]$-code meeting the Griesmer bound (1.3) if and only if C has a generator matrix whose columns correspond to all points in $\mathrm{PG}(k-1, q)$ except for the points of a disjoint union of three μ-flats in $\mathrm{PG}(k-1, q)$.*

2. Preliminary results

In order to prove Theorem 1.1, we shall prepare several results in this section. Since $1 < q/(q+1-h) < 2$ in the case $h = 3$ and $q \geqslant 5$, we have the following lemma from Theorem A.2.

Lemma 2.1. *There is no $\{3v_2, 3v_1;\ 2, q\}$-minihyper in the case $q \geqslant 5$.*

Lemma 2.2. *Let F be any $\{3v_2, 3v_1;\ t, q\}$-minihyper where $t \geqslant 3$ and $q \geqslant 5$.*

(1) *If H is a hyperplane in* $\mathrm{PG}(t,q)$ *such that* $m(q+1) \leqslant |F \cap H| < (m+1)(q+1)$ *for some integer m in* $\{0,1,2,3\}$, *then $F \cap H$ is an $\{f,m;\, t,q\}$-minihyper in H where* $f = |F \cap H|$.

(2) $|F \cap H| = 3, q+3, 2q+3$ *or* $3q+3$ *for any hyperplane H in* $\mathrm{PG}(t,q)$.

Proof. (1) Let H be a hyperplane in $\mathrm{PG}(t,q)$ such that $m(q+1) \leqslant |F \cap H| < (m+1)(q+1)$. In the case $m = 0$, it is obvious that (1) holds.

In the case $1 \leqslant m \leqslant 3$, suppose there exists a $(t-2)$-flat G in H such that $|F^* \cap G| \leqslant m-1$ where $F^* = F \cap H$. Let H_i $(i = 1,2,\ldots,q)$ be q hyperplanes in $\mathrm{PG}(t,q)$, except for H, which contain G. Then we have

$$|F| = |F \cap H| + \sum_{i=1}^{q} \{|F \cap H_i| - |F^* \cap G|\} \geqslant m(q+1)$$
$$+ q\{3-(m-1)\} > 3(q+1) = |F|,$$

which is a contradiction. Hence $|F^* \cap G| \geqslant m$ for every $(t-2)$-flat G in H. If $|F^* \cap G| = m$ for some $(t-2)$-flat G in H, it follows that F^* is an $\{f,m;\, t,q\}$-minihyper in H.

Suppose $|F^* \cap G| \geqslant m+1$ for any $(t-2)$-flat G in H. Then there exists a subset K of the set F^* such that $|K \cap G| \geqslant m+1$ for every $(t-2)$-flat G in H and $|K \cap G_0| = m+1$ for some $(t-2)$-flat G_0 in H. Since $t \geqslant 3$ and $m+1 \leqslant 4 < q+1$, there exists a $(t-3)$-flat Δ in G_0 such that $K \cap \Delta = \emptyset$. Let G_i $(i = 1,2,\ldots,q)$ be q $(t-2)$-flats in the $(t-1)$-flat H, except for G_0, which contain Δ. Then

$$|F^*| \geqslant |K| = \sum_{i=0}^{q} |K \cap G_i| \geqslant (m+1)(q+1) > |F^*|,$$

which is a contradiction. Hence, $|F^* \cap G| = m$ for some $(t-2)$-flat G in H.

(2) Suppose there exists a hyperplane H in $\mathrm{PG}(t,q)$ such that $mq+3 < |F \cap H| < (m+1)(q+1)$ for some integer m in $\{0,1,2\}$. Then it follows from (1) that there exists a $(t-2)$-flat G in H such that $|F \cap G| = m$. Let H_i $(i = 1,2,\ldots,q)$ be q hyperplanes in $\mathrm{PG}(t,q)$, except for H, which contain G. Then

$$|F| = |F \cap H| + \sum_{i=1}^{q} \{|F \cap H_i| - |F \cap G|\} > 3(q+1) = |F|,$$

which is a contradiction. Hence, $|F \cap H| = 3$, $q+1$, $q+2$, $q+3$, $2q+2$, $2q+3$ or $3q+3$ for any hyperplane H in $\mathrm{PG}(t,q)$.

Suppose there exists a hyperplane H in $\mathrm{PG}(t,q)$ such that $|F \cap H| = q+1, q+2$, or $2q+2$.

Case I: ($|F \cap H| = q+1$). It follows from (1), $v_1 = 1$ and $v_2 = q+1$ that $F \cap H$ is a $\{v_2, v_1; t,q\}$-minihyper in H. Hence, it follows from Theorem A.3 ($\lambda = 1$) that $F \cap H$ is a 1-flat (denoted by L) in H. Let G be any $(t-2)$-flat in H such that $|G \cap L| = 1$ (i.e., $|G \cap F| = 1$) and let H_i $(i = 1,2,\ldots,q)$ be q hyperplanes in $\mathrm{PG}(t,q)$, except for H, which contain G. Without loss of generality, we can assume that $|F \cap H_1| \geqslant |F \cap H_2| \geqslant \cdots \geqslant |F \cap H_q|$.

Since $\sum_{i=1}^{q} |F \cap (H_i \backslash G)| = |F| - |F \cap H| = 2q+2$ and $|F \cap (H_i \backslash G)| = |F \cap H_i| - |F \cap G| \geqslant 2$ for $i = 1, 2, \ldots, q$, it follows that $|F \cap (H_1 \backslash G)| = 3$ or 4. Since $|F \cap H_1| = |F \cap G| + |F \cap (H_1 \backslash G)|$ and $|F \cap G| = 1$, this implies that $|F \cap H_1| = 4$ or 5, i.e., $3 < |F \cap H_1| < q+1$ in the case $q \geqslant 5$, which is a contradiction. Hence, there is no hyperplane H in $\mathrm{PG}(t,q)$ such that $|F \cap H| = q+1$.

Case II: ($|F \cap H| = q+2$). It follows from (1) and Theorem A.4 ($\lambda_1 = 0$, $\lambda_2 = 1$) that $F \cap H = L \cup \{P\}$ for some 1-flat L and some point P in H. Let G be any $(t-2)$-flat in H such that $|G \cap L| = 1$ and $P \notin G$ (i.e., $|G \cap F| = 1$) and let H_i ($i = 1, 2, \ldots, q$) be q hyperplanes in $\mathrm{PG}(t,q)$, except for H, which contain G where $|F \cap H_1| \geqslant |F \cap H_2| \geqslant \cdots \geqslant |F \cap H_q|$. Then $|F \cap H_1| = 4$, which is a contradiction.

Case III: ($|F \cap H| = 2q+2$). It follows from (1) and Theorem A.4 ($\lambda_1 = \lambda_2 = 1$) that (a) in the case $t = 3$ (i.e., $t-1 = 2$), there is no 2-flat H in $\mathrm{PG}(3,q)$ such that $|F \cap H| = 2(q+1)$ and (b) in the case $t \geqslant 4$, $F \cap H = L_1 \cup L_2$ for some 1-flats L_1 and L_2 in H which are mutually disjoint.

In the case (b), let G be a $(t-2)$-flat in H such that $|G \cap L_1| = 1$ and $|G \cap L_2| = 1$ (i.e., $|G \cap F| = 2$) and let H_i ($i = 1, 2, \ldots, q$) be q hyperplanes in $\mathrm{PG}(t,q)$, except for H, which contain G where $|F \cap H_1| \geqslant |F \cap H_2| \geqslant \cdots \geqslant |F \cap H_q|$. Then $|F \cap H_1| = 4$, which is a contradiction. This completes the proof. □

Lemma 2.3. *If F is a $\{3v_2, 3v_1; t, q\}$-minihyper in the case $t \geqslant 3$ and $q \geqslant 5$, then F is a union of three 1-flats in* $\mathrm{PG}(t,q)$ *which are mutually disjoint.*

Proof. Let F be any $\{3v_2, 3v_1; t, q\}$-minihyper. There exists a hyperplane H in $\mathrm{PG}(t,q)$ such that $|F \cap H| = 3$, i.e., $F \cap H = \{P_1, P_2, P_3\}$ for some points P_1, P_2 and P_3 in H. Since $q+1 > 2$, there exists a $(t-2)$-flat Δ_l in H such that $\{P_1, P_2, P_3\} \cap \Delta_l = \{P_l\}$ for each integer l in $\{1, 2, 3\}$.

Let H_i ($i = 1, 2, \ldots, q$) be q hyperplanes in $\mathrm{PG}(t,q)$, except for H, which contain Δ_1, where $|F \cap H_1| \geqslant |F \cap H_2| \geqslant \cdots \geqslant |F \cap H_q|$. Since $\sum_{i=1}^{q} |F \cap (H_i \backslash \Delta_1)| = |F| - |F \cap H| = 3q$ and $|F \cap (H_i \backslash \Delta_1)| = |F \cap H_i| - |F \cap \Delta_1| \geqslant 2$ for $i = 1, 2, \ldots, q$, it follows that $3 \leqslant |F \cap (H_1 \backslash \Delta_1)| \leqslant 3q - 2(q-1) = q+2$, i.e., $4 \leqslant |F \cap H_1| = q+3$. Hence, it follows from Lemma 2.2 and Theorem A.5 that $|F \cap H_1| = q+3$ and $F \cap H_1 = L_1 \cup \{Q_1, Q_2\}$ for some 1-flat L_1 and some points Q_1 and Q_2 in H_1. Since $H \cap H_1 = \Delta_1$ and $L_1 \cap \Delta_1 = \{P_1\}$, this implies that there exists a 1-flat L_1 in F such that $\{P_1, P_2, P_3\} \cap L_1 = \{P_1\}$.

Similarly, it can be shown that there exists a 1-flat L_l in F such that $\{P_1, P_2, P_3\} \cap L_l = \{P_l\}$ for $l = 2, 3$. This implies that $F = L_1 \cup L_2 \cup L_3 \cup S$ for some set S in $\mathrm{PG}(t,q)$ such that $|S| = 3(q+1) - |L_1 \cup L_2 \cup L_3|$. If L_1, L_2 and L_3 are mutually disjoint, then $S = \emptyset$ and Lemma 2.3 holds.

Suppose L_1, L_2 and L_3 are not mutually disjoint. Without loss of generality, we can assume that $L_1 \cap L_2 \neq \emptyset$ (i.e., $L_1 \cap L_2 = \{Q\}$).

Case I: ($t = 3$). Let Π be the hyperplane (i.e., 2-flat) in PG$(3, q)$ which contains L_1 and L_2. Then $|F \cap \Pi| \geqslant 2q + 1$. Hence, it follows from Lemma 2.2 that $|F \cap \Pi| = 2q + 3$ or $3q + 3$.

In the case $|F \cap \Pi| = 2q + 3$, it follows from Lemma 2.2 that $F \cap \Pi$ is a $\{2q + 3, 2;\ 3, q\}$-minihyper in Π. Since Π is a 2-flat, this implies that there exists a $\{v_1 + 2v_2, v_0 + 2v_1; 2, q\}$-minihyper where $v_0 = 0$, $v_1 = 1$ and $v_2 = q + 1$. Hence, we have a contradiction from Theorem A.6.

In the case $|F \cap \Pi| = 3q + 3$, it follows from $|F| = 3q + 3$ that $F \subset \Pi$. This implies that there exists a $\{3v_2, 3v_1;\ 2, q\}$-minihyper which is contradictory to Lemma 2.1. Hence, L_1, L_2 and L_3 must be mutually disjoint.

Case II: ($t \geqslant 4$). Let Π be a hyperplane in PG(t, q) which contains L_1 and L_2. Since $|F \cap \Pi| \geqslant 2q + 1$, it follows from Lemma 2.2 that $|F \cap \Pi| = 2q + 3$ or $3q + 3$.

In the case $|F \cap \Pi| = 2q + 3$, it follows from Lemma 2.2 and Theorem A.6 that $F \cap \Pi$ is a union of one point and two 1-flats in Π which are mutually disjoint. Since $L_1 \cup L_2 \subset F \cap \Pi$ and $L_1 \cap L_2 \neq \emptyset$, this is a contradiction.

In the case $|F \cap \Pi| = 3q + 3$, it follows from $|F| = 3q + 3$ that F is a $\{3v_2, 3v_1;\ t, q\}$-minihyper in the $(t - 1)$-flat Π. Since Lemma 2.3 holds in the case $t = 3$ (cf. case I), it follows by induction on t that F is a union of three 1-flats in Π which are mutually disjoint. This completes the proof. $\square$

3. The proof of Theorem 1.1

In order to prove Theorem 1.1, we shall use the following lemma due to Hamada [3].

Lemma 3.1. *Let G be a $(t - 2)$-flat in* PG(t, q) *and let W_1, W_2 and W_3 be three $(\mu - 2)$-flats in G which are mutually disjoint where $4 \leqslant 2\mu \leqslant t$ and $q \geqslant 2$. Let H_i $(i = 0, 1, \ldots, q)$ be $q + 1$ hyperplanes in* PG(t, q) *which contain G. Let V_{ij} $(i = 0, 1, \ldots, q$, $j = 1, 2, 3)$ be $(\mu - 1)$-flats in H_i such that* (a) *$G \cap V_{ij} = W_j$ and* (b) *V_{i1}, V_{i2} and V_{i3} are mutually disjoint. Let $Y_j = \bigcup_{i=0}^{q} V_{ij}$ for $j = 1, 2, 3$. Then $Y_1 \cup Y_2 \cup Y_3$ is a $\{3v_{\mu+1}, 3v_\mu; t, q\}$-minihyper if and only if Y_1, Y_2 and Y_3 are three μ-flats in* PG(t, q) *which are mutually disjoint.*

Proof of Theorem 1.1. We shall prove Theorem 1.1 by induction on μ.

Case I: ($\mu = 1$). It follows from Remark 1.1 and Lemmas 2.1 and 2.3 that Theorem 1.1 holds.

Case II: ($\mu \geqslant 2$). Suppose there exists a $\{3v_{\mu+1}, 3v_\mu;\ t, q\}$-minihyper F for some integer $t > \mu$. Using a method similar to the proof of Lemma 2.2, it can be shown that $|F \cap G^*| \geqslant 3v_{\mu-1}$ for any $(t - 2)$-flat G^* in PG(t, q) and $|F \cap G| = 3v_{\mu-1}$ for some $(t - 2)$-flat G in PG(t, q).

Let H_i $(i = 0, 1, \ldots, q)$ be $q + 1$ hyperplanes in $\mathrm{PG}(t, q)$ which contain G. Since $|F \cap H_0| + \sum_{i=1}^{q}\{|F \cap H_i| - |F \cap G|\} = |F| = 3v_{\mu+1}$, it follows from $|F \cap H_i| \geqslant 3v_\mu$ $(i = 0, 1, \cdots, q)$ that $|F \cap H_i| = 3v_\mu$ for $i = 0, 1, \ldots, q$. Since $|F \cap G^*| \geqslant 3v_{\mu-1}$ for any $(t-2)$-flat G^* in H_i and $|F \cap G| = 3v_{\mu-1}$ for the $(t-2)$-flat G in H_i, this implies that $F \cap H_i$ is a $\{3v_\mu, 3v_{\mu-1}; t, q\}$-minihyper in the $(t-1)$-flat H_i for $i = 0, 1, \ldots, q$.

By induction on μ, it follows that (i) in the case $t - 1 \leqslant 2(\mu - 1)$ (i.e., $t \leqslant 2\mu - 1$), there is no $\{3v_\mu, 3v_{\mu-1}; t, q\}$-minihyper in H for any $(t-1)$-flat H, which is a contradiction, and (ii) in the case $t - 1 \geqslant 2(\mu - 1) + 1$ (i.e., $t \geqslant 2\mu \geqslant 4$), $F \cap H_i$ is a union of three $(\mu - 1)$-flats (denoted by V_{i1}, V_{i2} and V_{i3}) in H_i which are mutually disjoint. Hence, (i) in the case $t \leqslant 2\mu - 1$, there is no $\{3v_{\mu+1}, 3v_\mu; t, q\}$-minihyper and (ii) in the case $t \geqslant 2\mu$, $F \cap H_i = V_{i1} \cup V_{i2} \cup V_{i3}$ for some $(\mu - 1)$-flats V_{i1}, V_{i2} and V_{i3} in H_i which are mutually disjoint.

Let $W_j = G \cap V_{0j}$ for $j = 1, 2, 3$. Then W_j is a $(\mu - 2)$-flat or a $(\mu - 1)$-flat in G. If W_j is a $(\mu - 1)$-flat for some integer j, then $|W_j| = v_\mu$ and $|F \cap G| = |W_1| + |W_2| + |W_3| \geqslant v_\mu + 2v_{\mu-1} > 3v_{\mu-1} = |F \cap G|$, a contradiction. Hence, W_1, W_2, and W_3 must be $(\mu - 2)$-flats in G which are mutually disjoint.

Similarly, it can be shown that $G \cap V_{ij}$ is a $(\mu - 2)$-flat in G for $i = 1, 2, \ldots, q$ and $j = 1, 2, 3$. Since $G \cap V_{i1} = W_\alpha$, $G \cap V_{i2} = W_\beta$ and $G \cap V_{i3} = W_\gamma$ for some integers α, β and γ such that $\{\alpha, \beta, \gamma\} = \{1, 2, 3\}$, we can assume, without loss of generality, that $G \cap V_{ij} = W_j$ for $i = 1, 2, \ldots, q$ and $j = 1, 2, 3$.

Let $Y_j = \bigcup_{i=0}^{q} V_{ij}$ for $j = 1, 2, 3$. Then $F = Y_1 \cup Y_2 \cup Y_3$. Hence, it follows from Lemma 3.1 that F is a union of three μ-flats Y_1, Y_2 and Y_3 in $\mathrm{PG}(t, q)$ which are mutually disjoint. Since $\mathscr{F}(\mu, \mu, \mu; t, q) \neq \emptyset$ if and only if $t \geqslant 2\mu + 1$ (cf. Remark 1.1), it follows from Remark 1.1 that Theorem 1.1 holds in the case $\mu \geqslant 2$. This completes the proof. □

Appendix: Connections between minihypers and codes

Let $S(k, q)$ be the set of all column vectors c, $c = (c_1, c_2, \ldots, c_k)^{\mathrm{T}}$, in $W(k, q)$ such that either $c_1 = 1$ or $c_1 = c_2 = \cdots = c_{i-1} = 0$, $c_i = 1$ for some integer i in $\{2, 3, \ldots, k\}$ where $k \geqslant 3$ and $W(k, q)$ denotes a k-dimensional vector space consisting of column vectors over $\mathrm{GF}(q)$. Then $S(k, q)$ consists of all $(q^k - 1)/(q - 1)$ projectively distinct nonzero vectors in $W(k, q)$ which may be regarded as $(q^k - 1)/(q - 1)$ points in $\mathrm{PG}(k - 1, q)$.

Theorem A.1 (Hamada [3]). *Let F be a set of f vectors in $S(k, q)$ and let C be the subspace of $V(n, q)$ generated by a $k \times n$ matrix (denoted by G) whose column vectors are all the vectors in $S(k, q) \setminus F$ where $n = v_k - f$, $1 \leqslant f < v_k - 1$ and $v_i = (q^i - 1)/(q - 1)$ for any integer $i \geqslant 0$.*

(1) *Let $H_z = \{y \in S(k, q) \mid z \cdot y = 0$ over $\mathrm{GF}(q)\}$ for a nonzero vector z in $W(k, q)$. Then H_z is a hyperplane in $\mathrm{PG}(k - 1, q)$ and the weight of the code vector $z^{\mathrm{T}} G$ in C is equal to $|F \cap H_z| - v_{k-1} + n$ where z^{T} denotes the transpose of the vector z.*

(2) *In the case $k \geqslant 3$ and $1 \leqslant d < q^{k-1}$, C is an $[n, k, d; q]$-code meeting the Griesmer bound if and only if F is a $\{v_k - n, v_{k-1} - n + d; k - 1, q\}$-minihyper.*

Definition A.1. Two $[n, k, d; q]$-codes C_1 and C_2 are said to be equivalent if there exists a $k \times n$ generator matrix G_2 of the code C_2 such that $G_2 = G_1PD$ (or $G_2 = G_1DP$) for some permutation matrix P and some nonsingular diagonal matrix D with entries from $\mathrm{GF}(q)$, where G_1 is a $k \times n$ generator matrix of C_1.

From Theorem A.1 and Definition A.1, we have the following corollary.

Corollary A.1. *In the case $k \geqslant 3$ and $d = q^{k-1} - \sum_{i=1}^{h} q^{\lambda_i}$, there is a one-to-one correspondence between the set of all nonequivalent $[n, k, d; q]$-codes meeting the Griesmer bound and the set of all $\{\sum_{i=1}^{h} v_{\lambda_i+1}, \sum_{i=1}^{h} v_{\lambda_i}; k - 1, q\}$-minihypers where $n = v_k - \sum_{i=1}^{h} v_{\lambda_i+1}$.*

It is well known that in the special case $t = 2$, $2 \leqslant h < q$ and $\lambda_1 = \lambda_2 = \cdots = \lambda_h = 1$, the following theorem holds.

Theorem A.2. *If there exists an $\{hv_2, hv_1; 2, q\}$-minihyper F for some prime power q and some positive integer $h(< q)$, then $|F \cap H| = h$ or $q + 1$ for any 1-flat H in $\mathrm{PG}(2, q)$ and $q/(q + 1 - h)$ must be an integer where $v_1 = 1$ and $v_2 = q + 1$.*

The following four theorems due to Tamari and Hamada play an important role in proving Theorem 1.1.

Theorem A.3 (Tamari [13, 14]). *In the case $1 \leqslant \lambda < t$ and $t \geqslant 2$, F is a $\{v_{\lambda+1}, v_\lambda; t, q\}$-minihyper if and only if F is a λ-flat in $\mathrm{PG}(t, q)$.*

Theorem A.4 (Hamada [2, 5]). *Let t, λ_1 and λ_2 be integers such that $0 \leqslant \lambda_1 \leqslant \lambda_2 < t$ and let q be a prime power $\geqslant 3$.*

(1) *In the case $t \geqslant \lambda_1 + \lambda_2 + 1$, F is a $\{v_{\lambda_1+1} + v_{\lambda_2+1}, v_{\lambda_1} + v_{\lambda_2}; t, q\}$-minihyper if and only if F is a union of a λ_1-flat and a λ_2-flat in $\mathrm{PG}(t, q)$ which are mutually disjoint.*

(2) *In the case $t \leqslant \lambda_1 + \lambda_2$, there is no $\{v_{\lambda_1+1} + v_{\lambda_2+1}, v_{\lambda_1} + v_{\lambda_2}; t, q\}$-minihyper.*

Theorem A.5 (Hamada [4]). *In the case $t \geqslant 2$ and $q \geqslant 5$, F is a $\{2v_1 + v_2, 2v_0 + v_1; t, q\}$-minihyper if and only if F is a union of two points and one 1-flat in $\mathrm{PG}(t, q)$ which are mutually disjoint.*

Theorem A.6 (Hamada [5]). *Let q be a prime power $\geqslant 4$.*

(1) *In the case $t \geqslant 3$, F is a $\{v_1 + 2v_2, v_0 + 2v_1; t, q\}$-minihyper if and only if F is a union of one point and two 1-flats in $\mathrm{PG}(t, q)$ which are mutually disjoint.*

(2) *In the case $t = 2$, there is no $\{v_1 + 2v_2, v_0 + 2v_1; t, q\}$-minihyper.*

Acknowledgements

This research was partially supported by the Norwegian Research Council. The first author was partially supported by Grant-in-Aid for Scientific Research for the ministry of Education, Science and Culture under contract Numbers 403-4005-02640182.

References

[1] J.H. Griesmer, A bound for error-correcting codes, IBM J. Res. Develop. 4 (1960) 532–542.

[2] N. Hamada, Characterization resp. nonexistence of certain q-ary linear codes attaining the Griesmer bound, Bull. Osaka Women's Univ. 22 (1985) 1–47.

[3] N. Hamada, Characterization of minihypers in a finite projective geometry and its applications to error-correcting codes, Bull. Osaka Women's Univ. 24 (1987) 1–24.

[4] N. Hamada, Characterization of $\{(q+1)+2, 1; t, q\}$-minihypers and $\{2(q+1)+2, 2; 2, q\}$-minihypers in a finite projective geometry, Graphs Combin. 5 (1989) 63–81.

[5] N. Hamada, A characterization of some $[n, k, d; q]$-codes meeting the Griesmer bound using a minihyper in a finite projective geometry. Discrete Math. 116 (1993) 229–268.

[6] N. Hamada and M. Deza, A characterization of some $(n, k, d; q)$-codes meeting the Griesmer bound for given integers $k \geqslant 3$, $q \geqslant 5$ and $d = q^{k-1} - q^{\alpha} - q^{\beta} - q^{\gamma}$ $(0 \leqslant \alpha \leqslant \beta < \gamma < k-1$ or $0 \leqslant \alpha < \beta \leqslant \gamma < k-1)$, Bull. Inst. Math. Acad. Sinica 16 (1988) 321–338.

[7] N. Hamada and M. Deza, A characterization of $\{v_{\mu+1} + \varepsilon, v_{\mu}; t, q\}$-minihypers and its applications to error-correcting codes and factorial designs, J. Statist. Plann. Inference 22 (1989) 323–336.

[8] N. Hamada and T. Helleseth, A characterization of some minihypers in a finite projective geometry PG$(t, 4)$, European J. Combin. 11 (1990) 541–548.

[9] N. Hamada and T. Helleseth, A characterization of some $\{v_2 + 2v_3, v_1 + 2v_2; k-1, 3\}$-minihypers and some $(v_k - 30, k, 3^{k-1} - 21; 3)$-codes meeting the Griesmer bound, J. Statist. Plann. Inference 34 (1993) 387–402.

[10] N. Hamada and F. Tamari, Construction of optimal linear codes using flats and spreads in a finite projective geometry, European J. Combin. 3 (1982) 129–141.

[11] T. Helleseth, A characterization of codes meeting the Griesmer bound, Inform. and Control 50 (1981) 128–159.

[12] G. Solomon and J.J. Stiffler, Algebraically punctured cyclic codes, Inform. and Control 8 (1965) 170–179.

[13] F. Tamari, A note on the construction of optimal linear codes, J. Statist. Plann. Inference 5 (1981) 405–411.

[14] F. Tamari, On linear codes which attain the Solomon–Stiffler bound, Discrete Math. 49 (1984) 179–191.